Asymptotics for eigenelements of Laplacian in a domain with oscillating boundary

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Consider the following spectral problem :

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon \quad \text{in } \Omega^\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega^\varepsilon \backslash \Gamma_\varepsilon, \end{cases} \tag{1}$$

where Ω^ε is a domain with oscillating boundary, Γ_ε is the oscillating boundary, and ν denotes the outward unit normal vector to Ω^ε.

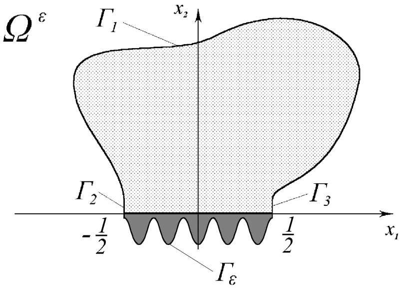

FIGURE – Membrane with oscillating boundary.

The domain Ω^ε is a small perturbation of a bounded domain Ω of $\mathbb{R}^2$ located in the upper half space. We assume the boundary $\partial\Omega$ (of Ω) to be piecewise smooth, consisting of four parts : $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where Γ_0 is the segment $[-\frac{1}{2}, \frac{1}{2}]$ on the abscissa axis, and Γ_2 and Γ_3 belong to the straight lines $x_1 = -\frac{1}{2}$ et $x_1 = \frac{1}{2}$, respectively.

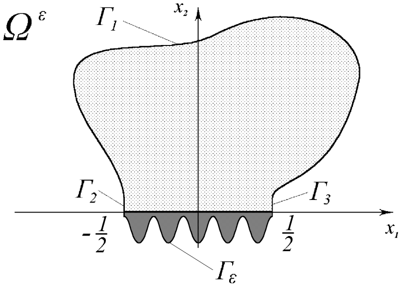

FIGURE – Membrane with oscillating boundary.

Here $\varepsilon = \frac{1}{2\mathcal{N}+1}$ is a small parameter where $\mathcal{N}$ is a large integer number. Given a smooth negative 1-periodic even function $F(\xi_1)$ such that $F'(\xi_1) = 0$ for $\xi_1 = \pm\frac{1}{2}$ and $\xi_1 = 0$, we denote

$$\Pi_\varepsilon = \{x \in \mathbb{R}^2 \ : \ x_1 \in (-\frac{1}{2}, \frac{1}{2}), \ \varepsilon F\left(\frac{x_1}{\varepsilon}\right) < x_2 \leq 0\}$$

and we define the domain Ω^ε by (see Figure 1)

$$\Omega^\varepsilon = \Omega \cup \Pi_\varepsilon.$$

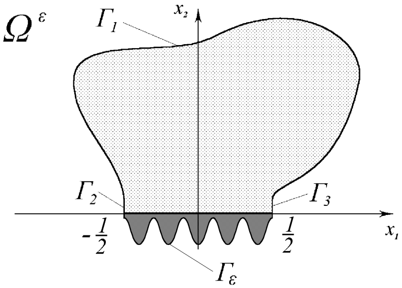

FIGURE – Membrane with oscillating boundary.

Denote

$$\Gamma = \{\xi \in \mathbb{R}^2 \ : \ -\frac{1}{2} < \xi_1 < \frac{1}{2}, \ \xi_2 = F(\xi_1)\}$$

and

$$\Pi = \{\xi \in \mathbb{R}^2 \ : \ -\frac{1}{2} < \xi_1 < \frac{1}{2}, \ \xi_2 > F(\xi_1)\}$$

(Π is a semi-infinite strip, see Figure).

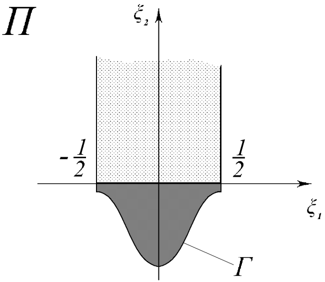

FIGURE – Cell of periodicity.

Thus the boundary of Ω^ε consists of four parts : $\partial\Omega^\varepsilon = \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}$, où

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^2 \ : \ (x_1, 0) \in \Gamma_0, \ x_2 = \varepsilon F\left(\frac{x_1}{\varepsilon}\right)\},$$

$$\Gamma_{2,\varepsilon} = \Gamma_2 \cup \{x \in \mathbb{R}^2 : x_1 = -\frac{1}{2}, \ \varepsilon F\left(-\frac{1}{2\varepsilon}\right) \leq x_2 \leq 0\},$$

$$\Gamma_{3,\varepsilon} = \Gamma_3 \cup \{x \in \mathbb{R}^2 \ : \ x_1 = \frac{1}{2}, \ \varepsilon F\left(\frac{1}{2\varepsilon}\right) \leq x_2 \leq 0\}.$$

We are interested in the asymptotic behavior of λ_ε and u_ε when $\varepsilon \to 0$.
We distinguish two cases :

- **λ_0 is a simple eigenvalue of the limit problem**
- **λ_0 is a multiple eigenvalue of the limit problem**

Theorem. Assume that λ_0 is a simple eigenvalue of the problem

$$\begin{cases} -\Delta u_0 = \lambda_0 u_0 \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \Gamma_0, \qquad \frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases} \tag{2}$$

and u_0 is the corresponding eigenfunction, with norm 1 in $L_2(\Omega)$. Then
a) there exists a simple eigenvalue λ_ε of the perturbed problem (1), converging to λ_0 as $\varepsilon \to 0$;
b) the asymptotic expansion at order 1 of λ_ε is

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + o(\varepsilon), \text{ with}$$

$$\lambda_1 = -C(F) \int\limits_{\Gamma_0} \left(\frac{\partial u_0}{\partial \nu} \right)^2 ds,$$

where $C(F)$ is a positive constant depending only on the function F, defined via the solution of

$$\begin{cases} \Delta_\xi X = 0 \quad \text{in } \Pi, \\ X = 0 \quad \text{on } \Gamma, \qquad \frac{\partial X}{\partial \xi_1} = 0 \quad \text{for } \xi_1 = \pm\frac{1}{2}, \\ X(\xi) = \xi_2 + C(F) \quad \text{for } \xi_2 \to +\infty. \end{cases} \tag{3}$$

To prove the result we employ the method of matching of asymptotic expansions, initiated by A.M. Il'in (1976), ... in different problems.

We consider an asymptotic expansion inside the domain Ω (called *external* expansion) in the form

$$u_\varepsilon(x) = u_0(x) + \dots$$

and an asymptotic expansion of the eigenvalue λ_ε

$$\lambda_\varepsilon = \lambda_0 + \dots .$$

Since u_0 is not defined in a neighborhood of Γ_ε, we introduce an expansion (called *inner* expansion) in a neighborhood of Γ_ε, then we use truncation functions to build an asymptotic expansion in the whole domain Ω^ε.

Consider the Taylor expansion of u_0 (with respect to x_2) as $x_2 \to 0$. According to the limit problem (2) verified by u_0, we have

$$u_0(x) = \alpha_0(x_1)x_2 + O(x_2^3),$$

where $\alpha_0(x_1) = \frac{\partial u_0}{\partial x_2}\Big|_{x_2=0}$ and

$$\alpha_0'\left(\pm\frac{1}{2}\right) = 0.$$

By the change of variables $\xi_2 = \frac{x_2}{\varepsilon}$, we deduce that

$$u_0(x_1, \varepsilon\xi_2) = \varepsilon\alpha_0(x_1)\xi_2 + O(\varepsilon^3\xi_2^3).$$

By definition, the leading term of the inner asymptotic expansion satisfies the boundary conditions of the perturbed problem (1) on Γ_ε and admits the asymptotic expansion (as $\xi_2 \to +\infty$) as above.

Then the inner asymptotic expansion is in the form

$$u_\varepsilon(x) = \varepsilon v_1\left(\xi; x_1\right) + \dots$$

where $\xi = \frac{x}{\varepsilon}$,

$$v_1(\xi; x_1) \sim \alpha_0(x_1)\xi_2 \quad \text{quand } \xi_2 \to +\infty,$$

and x_1 plays the role of a "slow variable".
We have

$$\Delta\left(v_1(\frac{x}{\varepsilon}; x_1)\right) = \varepsilon^{-2}\Delta_\xi v_1 + 2\varepsilon^{-1}\frac{\partial^2 v_1}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_1}{\partial x_1^2},$$

$$\begin{cases} \dfrac{\partial}{\partial \nu} v_1(\dfrac{x}{\varepsilon}; x_1) = \varepsilon^{-1}\dfrac{\partial v_1}{\partial \xi_1} + \dfrac{\partial v_1}{\partial x_1} \quad \text{on } \Gamma_3, \\ \\ \dfrac{\partial}{\partial \nu} v_1(\dfrac{x}{\varepsilon}; x_1) = -\varepsilon^{-1}\dfrac{\partial v_1}{\partial \xi_1} - \dfrac{\partial v_1}{\partial x_1} \quad \text{on } \Gamma_2. \end{cases}$$

From the perturbed problem (1) we deduce, by identifying the terms of order ε^{-1} for the equation and ε^0 for the boundary conditions, the boundary-value problem :

$$\begin{cases} \Delta_\xi v_1 = 0 \quad \text{in } \Pi, \\ v_1 = 0 \quad \text{on } \Gamma, \qquad \frac{\partial v_1}{\partial \xi_1} = 0 \quad \text{for } \xi_1 = \pm\frac{1}{2}, \\ v_1 \sim \alpha_0(x_1)\xi_2 \quad \text{as } \xi_2 \to +\infty. \end{cases}$$

Thus,

$$v_1(\xi; x_1) = \alpha_0(x_1)X(\xi)$$

where X is the solution of (3) in the semi-infinite strip, and then

$$v_1(\xi; x_1) = \alpha_0(x_1)(\xi_2 + C(F)) \quad \text{as } \xi_2 \to +\infty.$$

Since

$$\frac{\partial v_1}{\partial x_1}(\xi; x_1) = 0 \quad \text{for } x_1 = \pm\frac{1}{2},$$

we have

$$\frac{\partial v_1}{\partial \nu}\left(\frac{x}{\varepsilon}; x_1\right) = 0 \quad \text{on } \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}.$$

The inner expansion produced a discrepancy at infinity by the term : $\varepsilon C(F)\alpha_0(x_1)$.
Introducing a new term in the external expansion at order ε, we eliminate this discrepancy.
Rewriting the asymptotic of εv_1 as $\xi_2 \to +\infty$ in terms of the variable x, we see that the external expansion must have the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \dots,$$

where

$$u_1(x) \sim C(F)\alpha_0(x_1) \quad \text{quand } x_2 \to 0.$$

Since u_1 is smooth, it is equivalent to set :

$$u_1(x_1, 0) = C(F)\alpha_0(x_1).$$

Hence the boundary condition for u_1 on Γ_0.

Then we write

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \dots$$

Reporting in the perturbed problem (1), identifying the terms of oder ε^1, we deduce the boundary-value problem for u_1 :

$$\begin{cases} -\Delta u_1 = \lambda_0 u_1 + \lambda_1 u_0 \quad \text{dans } \Omega, \\ u_1 = C(F)\alpha_0 \quad \text{on } \Gamma_0, \\ \frac{\partial u_1}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

The constant λ_1 may be obtained from the solvability condition of this problem. Multiplying the previous equation by u_0 and taking into account of the normalisation in $L_2(\Omega)$, it follows that

$$-\int\limits_{\Omega} \Delta u_1 u_0 \, dx = \lambda_0 \int\limits_{\Omega} u_1 u_0 \, dx + \lambda_1.$$

Using twice the Green formula it follows that

$$-\int\limits_{\Omega} \Delta u_1 u_0 \, dx = \int\limits_{\Omega} (\nabla u_1, \nabla u_0) \, dx - \int\limits_{\partial\Omega} \frac{\partial u_1}{\partial \nu} u_0 \, ds$$

$$= -\int\limits_{\Omega} u_1 \Delta u_0 \, dx - \int\limits_{\partial\Omega} \frac{\partial u_1}{\partial \nu} u_0 \, ds + \int\limits_{\partial\Omega} \frac{\partial u_0}{\partial \nu} u_1 \, ds$$

$$= \lambda_0 \int\limits_{\Omega} u_1 u_0 \; dx - C(F) \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_0^2(x_1) \; dx_1.$$

Note that we used

$$\frac{\partial u_0}{\partial \nu} = -\frac{\partial u_0}{\partial x_2} = -\alpha_0(x_1), \text{ on } \Gamma_0.$$

It follows that

$$\lambda_1 = -C(F) \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_0^2(x_1) \; dx_1.$$

To define uniquely u_1 we impose

$$\int\limits_{\Omega} u_0(x) u_1(x) \; dx = 0.$$

Thus, the method of matching of asymptotic expansions allows to obtain the first order corrector of the eigenvalue of the perturbed problem (1). To justify the construction we continue the process.

Consider the Taylor expansion of u_1 (with respect to x_2) as $x_2 \to 0$. Since $u_1(x)$ is smooth, one can write

$$u_1(x) = u_1(x_1, 0) + \frac{\partial u_1}{\partial x_2}(x_1, 0)x_2 + \dots$$

then

$$u_1(x) = C(F)\alpha_0(x_1) + \alpha_1(x_1)x_2 + \dots,$$

where $\alpha_1(x_1) = \frac{\partial u_1}{\partial x_2}(x_1, 0)$. According to the regularity of u_1 homogeneous Neumann boundary conditions on Γ_2 and Γ_3 satisfied by u_1, we have

$$\alpha_1' \left(\pm \frac{1}{2} \right) = 0.$$

The construction of the external expansion implied a discrepancy of the asymptotics at 0 by the term : $\varepsilon^2 \alpha_1(x_1)$.
introdicing a new term of order 2 in the inner expansion, we eliminate this discrepancy.

Precisely, rewriting the asymptotic $u_0 + \varepsilon u_1$ (as $x_2 \to 0$) by setting $x_2 = \varepsilon \xi_2$, we deduce that the inner expansion must be in the form

$$u_\varepsilon(x) = \varepsilon v_1(\xi; x_1) + \varepsilon^2 v_2(\xi; x_1) + \dots,$$

where

$$v_2(\xi; x_1) \sim \alpha_1(x_1)\xi_2 \quad \text{quand } \xi_2 \to +\infty.$$

Reporting the inner expansion of u_ε in the perturbed problem (1) and identifying the terms of order ε^0 for the equation and ε the boundary conditions, we obtain :

$$\begin{cases} -\Delta_\xi v_2 = 2\dfrac{\partial^2 v_1}{\partial x_1 \partial \xi_1} \quad \text{in } \Pi, \\ \\ v_2 = 0 \quad \text{on } \Gamma, \qquad \dfrac{\partial v_2}{\partial \xi_1} = 0 \quad \text{for } \xi_1 = \pm\dfrac{1}{2}. \end{cases}$$

Consider the auxiliary problem in semi-infinite strip Π :

$$\Delta_\xi \widetilde{X} = \frac{\partial X}{\partial \xi_1} \quad \text{in } \Pi, \qquad \widetilde{X} = 0 \quad \text{on } \partial\Pi.$$

We show that this problem has a solution with the asymptotic

$$\widetilde{X}(\xi) = 0 \quad \text{quand } \xi_2 \to +\infty.$$

Note that,due to the eveness of F, the solution $\widetilde{X}$ is even in ξ_1, and then admits a 1-periodic extension with respect to ξ_1.
Then it is easy to see that the function

$$v_2(\xi; x_1) = \alpha_1(x_1) X(\xi) - 2\alpha_0'(x_1)\widetilde{X}(\xi)$$

is the 1-periodic solution, which admits the asymptotics

$$v_2(\xi; x_1) = \alpha_1(x_1)\xi_2 + C(F)\alpha_1(x_1) \quad \text{quand } \xi_2 \to +\infty.$$

Moreover we verify that

$$\frac{\partial v_2}{\partial \nu}\left(\frac{x}{\varepsilon}; x_1\right) = 0 \quad \text{on } \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}.$$

Denote by $\chi(s)$ a truncation function, $\chi(s) = 0$ for $s < 1$ and $\chi(s) = 1$ for $s > 2$. Define also $\widetilde{\chi}_t(x_2) = \chi\left(\frac{x_2}{t}\right)$, which equals 0 for $x_2 < t$ and 1 for $x_2 > 2t$, where $t > 0$ is sufficiently large.
Set

$$\widetilde{\lambda}_\varepsilon = \lambda_0 + \varepsilon\lambda_1,$$

and

$$\widetilde{u}_\varepsilon(x) = \left(u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x)\right)\chi\left(\frac{x_2}{\varepsilon^\beta}\right) + \left(\varepsilon v_1\left(\frac{x}{\varepsilon}; x_1\right) + \varepsilon^2 v_2\left(\frac{x}{\varepsilon}; x_1\right)\right)\left(1 - \chi\left(\frac{x_2}{\varepsilon^\beta}\right)\right),$$

where β is a fixed real number $(0 < \beta < 1)$, and

$$u_2(x) = C(F)\alpha_1(x_1)(1 - \widetilde{\chi}_t(x_2)).$$

We verify that :

$$\widetilde{u}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \quad \frac{\partial \widetilde{u}_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}.$$

Moreover, one can write

$$-\Delta \widetilde{u}_\varepsilon = \widetilde{\lambda}_\varepsilon \widetilde{u}_\varepsilon + f_\varepsilon \quad \text{in } \Omega^\varepsilon,$$

with, choosing $\frac{2}{3} < \beta < 1$,

$$\|f_\varepsilon\|_{L_2(\Omega^\varepsilon)} = o(\varepsilon).$$

Note that we also have

$$\|\widetilde{u}_\varepsilon\|_{L_2(\Omega^\varepsilon)} = 1 + o(1).$$

To conclude we use the following lemma.

Lemma. Consider the boundary-value problem, with $F_\varepsilon \in L^2(\Omega^\varepsilon)$:

$$\begin{cases} -\Delta U_\varepsilon = \lambda U_\varepsilon + F_\varepsilon & \text{in } \Omega^\varepsilon, \\ U_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, & \frac{\partial U_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}. \end{cases} \tag{4}$$

Assume that λ_0 is a eigenvalue of problem (2) of order $p \geq 1$. Then

(i) There are exactly p eigenvalues of the perturbed problem (1) converging to λ_0, as $\varepsilon \to 0$;

(ii) for λ proche de λ_0 the solution U_ε of problem (4) satisfies the estimate

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq \mathcal{C} \frac{\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}}{\prod_{j=1}^{p} |\lambda_\varepsilon^j - \lambda|}$$

where $\lambda_\varepsilon^1, \dots, \lambda_\varepsilon^p$ are the eigenvalue of problem (1), converging to λ_0 ;

(iii) if a solution U_ε of problem (4) is orthogonal in $L_2(\Omega^\varepsilon)$ to the eigenfunction u_ε^k of problem (1), corresponding to λ_ε^k, then

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq \mathcal{C} \frac{\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}}{\prod_{j=1; j\neq i}^{p} |\lambda_\varepsilon^j - \lambda|}.$$

Case where λ_0 is a multiple eigenvalue.

We assume, without loss of generality that λ_0 est double. Let $u_0^{(l)}$ $(l = 1, 2)$ the corresponding eigenfunctions, orthonormalized in $L_2(\Omega)$, i.e.

$$\begin{cases} -\Delta u_0^{(l)} = \lambda_0 u_0^{(l)} \ \text{ in } \Omega, \\ u_0^{(l)} = 0 \ \text{ on } \Gamma_0, \\ \frac{\partial u_0^{(l)}}{\partial \nu} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

$$\int\limits_{\Omega} (u_0^{(l)})^2 \, dx = 1, \quad \int_{\Omega} u_0^{(1)} \, u_0^{(2)} \, dx = 0, \quad l = 1, 2.$$

One can also impose :

$$\int\limits_{\Gamma_0} \frac{\partial u_0^{(1)}}{\partial \nu} \frac{\partial u_0^{(2)}}{\partial \nu} \, ds = 0.$$

Moreover, for simplicity we assume that

$$\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial u_0^{(1)}}{\partial x_2} \right)^2 dx_1 \neq \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial u_0^{(2)}}{\partial x_2} \right)^2 dx_1.$$

By the previous lemma there are two eigenvalues of problem (1), converging to λ_0, as $\varepsilon \to 0$. let us denote by $\lambda_\varepsilon^{(1)}$ and $\lambda_\varepsilon^{(2)}$ these eigenvalues ; the corresponding eigenfunctions orthonormalized in $L_2(\Omega_\varepsilon)$ are denoted $u_\varepsilon^{(l)}$ $(l = 1, 2)$.

Theorem. Under the previous assumptions and notations, the eigenvalues $\lambda_\varepsilon^{(l)}$ of problem (1), converging to λ_0 as $\varepsilon \to 0$, and the corresponding eigenfunctions $u_\varepsilon^{(l)}$ admit the expansions :

$$\lambda_\varepsilon^{(l)} = \lambda_0 + \varepsilon \lambda_1^{(l)} + o\left(\varepsilon^{\frac{5}{4} - \sigma}\right) \quad \text{for any } \sigma > 0,$$

$$\lambda_1^{(l)} = - C(F) \int\limits_{\Gamma_0} \left(\frac{\partial u_0^{(l)}}{\partial \nu} \right)^2 ds,$$

$$\| u_\varepsilon^{(l)} - u_0^{(l)} \|_{H^1(\Omega)} + \| u_\varepsilon^{(l)} \|_{H^1(\Omega^\varepsilon \setminus \overline{\Omega})} = o(1).$$

Remark. By the method of matching of asymptotic expansions one can generalize this theorem and establish the asymptotic expansions of $\lambda_\varepsilon^{(l)}$ and $u_\varepsilon^{(l)}$ at any order.

Formal construction of the asymptotics

We write the external expansion :

$$u_\varepsilon^{(l)}(x) = u_0^{(l)}(x) + \varepsilon u_1^{(l)}(x) + \varepsilon^2 u_2^{(l)}(x) + \varepsilon^3 u_3^{(l)}(x) + \sum_{i=4}^{\infty} \varepsilon^i u_i^{(l)}(x),$$

the series for the eigenvalues

$$\lambda_\varepsilon^{(l)} = \lambda_0 + \varepsilon \lambda_1^{(l)} + \varepsilon^2 \lambda_2^{(l)} + \varepsilon^3 \lambda_3^{(l)} + \sum_{i=4}^{\infty} \varepsilon^i \lambda_i^{(l)}$$

and the inner expansion :

$$u_\varepsilon^{(l)}(x) = \varepsilon v_1^{(l)}(\xi; x_1) + \varepsilon^2 v_2^{(l)}(\xi; x_1) + \varepsilon^3 v_3^{(l)}(\xi; x_1) + \sum_{i=4}^{\infty} \varepsilon^i v_i^{(l)}(\xi; x_1),$$

where $\xi = \frac{x}{\varepsilon}$.

Inserting these expansions in perturbed problem (1) and we find that the functions $u_i^{(l)}$ $(i = 1, 2, 3)$ satisfy the following boundary-value problems

$$\begin{cases} -\Delta u_1^{(l)} = \lambda_0 u_1^{(l)} + \lambda_1^{(l)} u_0^{(l)} \ \text{ in } \Omega, \\ \frac{\partial u_1^{(l)}}{\partial \nu} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

$$\begin{cases} -\Delta u_2^{(l)} = \lambda_0 u_2^{(l)} + \lambda_1^{(l)} u_1^{(l)} + \lambda_2^{(l)} u_0^{(l)} \ \text{ in } \Omega, \\ \frac{\partial u_2^{(l)}}{\partial \nu} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

$$\begin{cases} -\Delta u_3^{(l)} = \lambda_0 u_3^{(l)} + \lambda_1^{(l)} u_2^{(l)} + \lambda_2^{(l)} u_1^{(l)} + \lambda_3^{(l)} u_0^{(l)} \\ \text{in } \Omega, \\ \frac{\partial u_3^{(l)}}{\partial \nu} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

We complete these problems boundary conditions on Γ_0 in the form

$$u_i^{(l)} = \alpha_{i0}^{(l)} \ \text{ on } \Gamma_0, \quad i = 1, 2, \dots,$$

where $\alpha_{i0}^{(l)}(x_1)$ are unknown functions satisfying

$$\left. \frac{d^{2k+1} \alpha_{i0}^{(l)}}{d x_1^{2k+1}} \right|_{x_1 = \pm \frac{1}{2}} = 0, \quad k = 0, 1, \dots$$

Condition (23) is necessary for solvability of recurrent system of boundary value problems (23)–(23) in $C^\infty(\overline{\Omega})$. Moreover such solutions do exist if these problems are solvable in $H^1(\Omega)$, and in addition we deduce from the boundary value problems that the following formulae

$$\left.\frac{d^{2k+1}\alpha_{ij}^{(l)}}{dx_1^{2k+1}}\right|_{x_1=\pm\frac{1}{2}} = 0, \quad k = 0, 1, \dots,$$

are true, where

$$\alpha_{ij}^{(l)}(x_1) = \frac{1}{j!}\left.\frac{\partial^j u_i^{(l)}}{\partial x_2^j}\right|_{x_2=0}, \quad i, j = 0, 1, \dots,$$

Also it should be noted that it follows from Problem (2) that

$$\alpha_{02}^{(l)}(x_1) \equiv 0.$$

Note that if $\mathcal{F} \in H^1(\Omega)$ and $\alpha \in H^{1/2}(\Gamma_0)$ then for solvability in $H^1(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta u = \lambda_0 u + \mathcal{F} \ \text{ in } \Omega, \\ u = \alpha \ \text{ on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

it is necessary and sufficient to have the two identities

$$\int \mathcal{F} u_0^{(l)}\, dx = \int \alpha \frac{\partial u_0^{(l)}}{\partial \nu}\, ds, \quad l = 1, 2.$$

By analogous way we obtain the equations and boundary conditions satisfied by the functions $v_i^{(l)}$:

$$\begin{cases} \Delta_\xi v_1^{(l)} = 0 \ \text{ in } \Pi, \\ v_1^{(l)} = 0 \ \text{ on } \Gamma, \\ \dfrac{\partial v_1^{(l)}}{\partial \xi_1} = 0 \ \text{ for } \xi_1 = \pm\dfrac{1}{2}, \ x_1 = \pm\dfrac{1}{2}, \end{cases}$$

$$\begin{cases} -\Delta_\xi v_2^{(l)} = 2\dfrac{\partial^2 v_1^{(l)}}{\partial x_1 \partial \xi_1} \ \text{ in } \Pi, \\ v_2^{(l)} = 0 \ \text{ on } \Gamma, \\ \dfrac{\partial v_2^{(l)}}{\partial \xi_1} = -\dfrac{\partial v_1^{(l)}}{\partial x_1} \ \text{ for } \xi_1 = \pm\dfrac{1}{2}, \ x_1 = \pm\dfrac{1}{2}, \end{cases}$$

$$\begin{cases} -\Delta_\xi v_3^{(l)} = 2\dfrac{\partial^2 v_2^{(l)}}{\partial x_1 \partial \xi_1} + \dfrac{\partial^2 v_1^{(l)}}{\partial x_1^2} + \lambda_0 v_1^{(l)} \ \text{ in } \Pi, \\ v_3^{(l)} = 0 \ \text{ on } \Gamma, \\ \dfrac{\partial v_3^{(l)}}{\partial \xi_1} = -\dfrac{\partial v_2^{(l)}}{\partial x_1} \ \text{ for } \xi_1 = \pm\dfrac{1}{2}, \ x_1 = \pm\dfrac{1}{2}. \end{cases}$$

We add the boundary conditions at infinity (as $\xi_2 \to +\infty$), by matching the inner and external expansions. We obtain

$$\sum_{i=0}^{3} \varepsilon^i u_i^{(l)}(x) = \sum_{i=1}^{3} \varepsilon^i V_i^{(l)}(\xi; x_1) + O\left(\varepsilon^4(\xi_2^4 + \xi_2)\right)$$
$$\text{as} \quad x_2 = \varepsilon\xi_2 \to 0,$$

where

$$V_1^{(l)} = \alpha_{01}^{(l)}(x_1)\xi_2 + \alpha_{10}^{(l)}(x_1),$$
$$V_2^{(l)} = \alpha_{11}^{(l)}(x_1)\xi_2 + \alpha_{20}^{(l)}(x_1),$$
$$V_3^{(l)} = \alpha_{03}^{(l)}(x_1)\xi_2^3 + \alpha_{12}^{(l)}(x_1)\xi_2^2 + \alpha_{21}^{(l)}(x_1)\xi_2 + \alpha_{30}^{(l)}(x_1).$$

We must find $\lambda_i^{(l)}$ et $\alpha_{i0}^{(l)}(x_1)$ so that the different boundary-value problems satisfied by $u_i^{(l)}$ and $v_i^{(l)}$ are solvable :

$$v_i^{(l)} \sim V_i^{(l)} \text{ as } \xi_2 \to +\infty.$$

Let us determine $\alpha_{10}^{(l)}(x_1)$ and $v_1^{(l)}(\xi;x_1)$. We verify that the function defined by

$$v_1^{(l)}(\xi;x_1)=\alpha_{01}^{(l)}(x_1)X(\xi)$$

is the 1-periodic solution of the boundary-value problem

$$\begin{cases} \Delta_\xi v_1^{(l)}=0 \text{ in } \Pi, \\ v_1^{(l)}=0 \text{ on } \Gamma, \\ \dfrac{\partial v_1^{(l)}}{\partial \xi_1}=0 \text{ pour } \xi_1=\pm\dfrac{1}{2},\ x_1=\pm\dfrac{1}{2}, \end{cases}$$

with the asymptotics

$$v_1^{(l)}(\xi;x_1)=\alpha_{01}^{(l)}(x_1)(\xi_2+C(F)) \text{ as } \xi_2\to+\infty.$$

Then, setting

$$\alpha_{10}^{(l)}(x_1)=C(F)\alpha_{01}^{(l)}(x_1),$$

we get that $v_1^{(l)}$ as defined above satisfies also

$$v_1^{(l)}\sim V_1^{(l)}=\alpha_{01}^{(l)}(x_1)\xi_2+\alpha_{10}^{(l)}(x_1) \text{ as } \xi_2\to+\infty.$$

Thus we constructed $\alpha_{10}^{(l)}$ et $v_1^{(l)}$.

Then we determine $\lambda_1^{(l)}$ et $u_1^{(l)}$.

$$\lambda_1^{(l)} = -C(F) \int_{-\frac{1}{2}}^{\frac{1}{2}} (\alpha_{01}^{(l)})^2(x_1)\ dx_1.$$

We choose $u_1^{(l)}$ in the form :

$$u_1^{(l)} = \widetilde{u}_1^{(l)} + \kappa_1^{(l)} u_0^{(l^*)},$$

where

$$\int_\Omega \widetilde{u}_1^{(l)}(x) u_0^{(k)}(x)\ dx = 0, \quad l, k = 1, 2$$

and the constants $\kappa_1^{(l)}$ are arbitrary (to be found so that $u_2^{(l)}$ exists). Here, $l^* = 1$ if $l = 2$ et $l^* = 2$ if $l = 1$. Thus,

$$\alpha_{11}^{(l)} = \widetilde{\alpha}_{11}^{(l)} + \kappa_1^{(l)} \alpha_{01}^{(l^*)},$$

where

$$\widetilde{\alpha}_{11}^{(l)} = \frac{\partial \widetilde{u}_1^{(l)}}{\partial x_2}\bigg|_{x_2=0}, \ \frac{d^{2k+1}\widetilde{\alpha}_{11}^{(l)}}{dx_1^{2k+1}}\bigg|_{x_1=\pm\frac{1}{2}} = 0, \ k = 0, 1, \ldots$$

Then we determine $\alpha_{20}^{(l)}(x_1)$ and $v_2(\xi; x_2)$. Let $\widetilde{X}$ be the solution of the problem, in the semi-infinite strip Π,

$$\begin{cases} \Delta_\xi \widetilde{X} = \frac{\partial X}{\partial \xi_1} \text{ in } \Pi, \\ \widetilde{X} = 0 \text{ on } \partial\Pi, \quad \widetilde{X}(\xi) = 0 \text{ quand } \xi_2 \to +\infty. \end{cases}$$

Then, the function defined by

$$v_2^{(l)}(\xi; x_1) = \alpha_{11}^{(l)}(x_1) X(\xi) - 2(\alpha_{01}^{(l)})'(x_1)\widetilde{X}(\xi)$$

is the 1-periodic solution of the boundary-value problem

$$\begin{cases} -\Delta_\xi v_2^{(l)} = 2\dfrac{\partial^2 v_1^{(l)}}{\partial x_1 \partial \xi_1} \text{ in } \Pi, \\ v_2^{(l)} = 0 \text{ on } \Gamma, \\ \dfrac{\partial v_2^{(l)}}{\partial \xi_1} = 0 \text{ for } \xi_1 = \pm\dfrac{1}{2},\ x_1 = \pm\dfrac{1}{2}, \end{cases}$$

$$v_2^{(l)}(\xi; x_1) = \alpha_{11}^{(l)}(x_1)\left(\xi_2 + C(F)\right) \text{ as } \xi_2 \to +\infty,$$

verifying also $v_2(l) \sim V_2^{(l)} = \alpha_{11}^{(l)}(x_1)\xi_2 + \alpha_{20}^{(l)}(x_1)$, if we choose

$$\alpha_{20}^{(l)}(x_1) = C(F)\left(\widetilde{\alpha}_{11}^{(l)} + \kappa_1^{(l)}\alpha_{01}^{(l^*)}\right).$$

Thus we defined $v_2^{(l)}$ and $\alpha_{20}^{(l)}$ (modulo $\kappa_1^{(l)}$, which is unknown yet).

Then we determine $\lambda_2^{(l)}$, $u_2^{(l)}$ and $\kappa_1^{(l)}$. We obtain

$$\lambda_2^{(l)} = -C(F) \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{\alpha}_{11}^{(l)}(x_1)\alpha_{01}^{(l)}(x_1)\ dx_1$$

then

$$\kappa_1^{(l)} = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{\alpha}_{11}^{(l)}(x_1)\alpha_{01}^{(l^*)}(x_1)\ dx_1 \right) \Bigg/ \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(\alpha_{01}^{(l)}\right)^2 (x_1) - \left(\alpha_{01}^{(l^*)}\right)^2 (x_1) \right)\ dx_1 \right).$$

Hence $u_1^{(l)}$. We choose $u_2^{(l)}$ in the form

$$u_2^{(l)} = \widetilde{u}_2^{(l)} + \kappa_2^{(l)} u_0^{(l^*)},$$

where

$$\int_{\Omega} \widetilde{u}_2^{(l)}(x)u_0^{(k)}(x)\ dx = 0, \quad l, k = 1, 2$$

and the constants $\kappa_2^{(l)}$ are arbitrary.
...ETC...