Effective boundary condition for Stokes flow over a very rough surface

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A joint work with

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We study the asymptotic behaviour of the solution of Stokes equations in a 3-dimensional domain with highly oscillating boundary.

Let $S = (0, l_1) \times (0, l_2)$, $\widetilde{S} = (a_1, b_1) \times (a_2, b_2)$, with $0 < a_i < b_i < l_i$ $(i = 1, 2)$ so that $\widetilde{S} \subset S$.
Let η_ε be the εS-periodic function defined on εS by

$$\eta_\varepsilon\left(x'\right) = \begin{cases} l_3 & \text{if } x' \in \varepsilon\left(S \backslash \widetilde{S}\right), \\ l_3' & \text{if } x' \in \varepsilon \widetilde{S}, \end{cases}$$

with $l_3 < l_3'$, $x' = (x_1, x_2)$, and ε is a small positive parameter. Let

$$\Omega_\varepsilon = \left\{ x = (x', x_3) \in \mathbb{R}^3 : \ x' \in S, \ b(x') < x_3 < \eta_\varepsilon(x') \right\}$$

where b is a smooth function on $\mathbb{R}^2$, S-periodic and such that $b(x') < l_3$ for every $x' \in \mathbb{R}^2$. We assume that $1/\varepsilon \in \mathbb{N}$.
The domain Ω_ε is bounded at the bottom by the smooth wall

$$P = \left\{ x = (x', x_3) \in \mathbb{R}^3 \ : \ x' \in S, \ x_3 = b(x') \right\}$$

and at the top by the rough wall

$$R_\varepsilon = \partial \Omega_\varepsilon \setminus \left(\overline{P \cup \{(x', x_3) \in \mathbb{R}^3 \ : \ x' \in \partial S, \ b(x') < x_3 < l_3\}}\right).$$

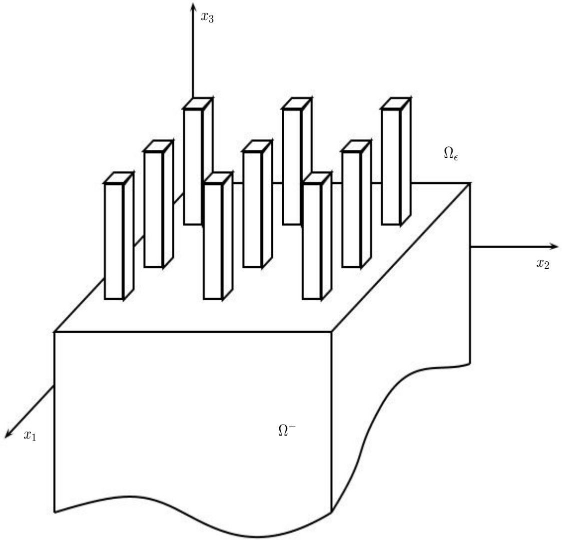

FIGURE – Domain Ω_ε

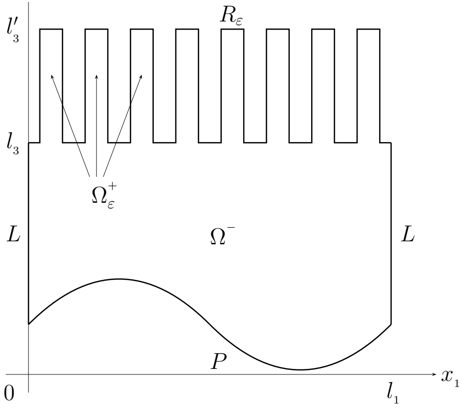

Figure 1: Vertical section of the domain Ω_ε

The velocity $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2}, u_{\varepsilon 3})$ and the pressure p_ε of the fluid satisfy

$$\begin{cases} -\nu \Delta u_\varepsilon + \nabla p_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } P \cup R_\varepsilon, \\ (u_\varepsilon, p_\varepsilon) \ S\text{-periodic } \ (\text{with respect to } x'), \end{cases} \tag{1}$$

where the source term f belongs to $\left(L^2(\Omega)\right)^3$, with

$$\Omega = \{(x', x_3) \in \mathbb{R}^3 \, : \, x' \in S, \ b(x') < x_3 < l_3'\},$$

representing the "limit domain", as ε tends to zero.

Our aim is to study the asymptotic behavior, as ε goes to 0, of the solution $(u_\varepsilon, p_\varepsilon)$ of (1) satisfying $\int_{\Omega^-} p_\varepsilon \, dx = 0$, where

$$\Omega^- = \left\{(x', x_3) \in \mathbb{R}^3 \, : \, x' \in S, \ b(x') < x_3 < l_3\right\}.$$

- Using boundary layer correctors, we construct an asymptotic approximation of the solution $(u_\varepsilon, p_\varepsilon)$ of (1) in Ω_ε.
- We derive an effective boundary condition of Navier's type, called wall law, for the Stokes system (1)

Some references

Formal derivation of wall laws (high Reynolds number) :

1. Y. Achdou, O. Pironneau. Domain decomposition and wall laws, CRAS, 1995.
2. Y. Achdou, O. Pironneau, F. Valentin. Effective boundary conditions for laminar flows over periodic rough boundaries, J. Comp. Phys., 1998.

Couette flow and applications to the drag reduction (small Reynolds number) :

3. Y. Amirat and J. Simon, Riblets and drag minimization, Contemp. Math., 209, AMS, 1997.
4. Y. Amirat, D. Bresch, J. Lemoine and J. Simon, Effect of rugosity on a flow governed by Navier-Stokes equations, Quarterly of Appl. Math., 2001.

Justification of the Navier's slip condition for laminar 2-D Poiseuille flow and 3-D Couette flow (moderate Reynolds number) :

5. W. Jäger and A. Mikelić, On the Roughness-induced effective boundary conditions for an incompressible viscous flow, J. Differential Equations, 2001.
6. W. Jäger and A. Mikelić, Couette flows over a rough boundary and drag reduction, Comm. Math. Phys., 2003.

Flows governed by Navier-Stokes equations together with Navier's law on a rough surface :

7. J. Casado-Díaz, E. Fernández-Cara, J. Simon, Why viscous fluids adhere to rugose walls (a mathematical explanation), J. Differential Equations, 2003.
8. D. Bucur, E. Feireisl, Š. Nečasová, J. Wolf, On the asymptotic limit of the Navier-Stokes system with rough boundaries, J. Differential equations, 2008.

Flows governed by Navier-Stokes equations over a boundary with random roughness :

9. A. Basson, D. Géerard-Varet, Wall laws for fuid flows at a boundary with random roughness, Comm. Pure Appl. Math. 61 (2008), no. 7, 941–987.

In these works the amplitude and the frequency of the oscillations are of the same order ε. **The present work deals with the case with highly oscillating boundary : The amplitude of the oscillations is fixed and the frequency is of order ε.**

A convergence result

We assume that the function b is Lipschitz-continuous. Denote

$$\Omega_\varepsilon^+ = \{(x', x_3) \in \Omega_\varepsilon : l_3 < x_3 < l_3'\}, \quad \Sigma = S \times \{l_3\}.$$

For each $m \geq 0$, we introduce the space

$$H^m_{\mathrm{per}}(\Omega_\varepsilon) = \{v \in H^1(A) \text{ for any bounded open set } A \subset \mathcal{O}_\varepsilon,$$
$$v(x + (l_1, 0, 0)) = v(x + (0, l_2, 0)) = v(x) \text{ for a.e. } x \in \mathcal{O}_\varepsilon\}$$

where $\mathcal{O}_\varepsilon = \left\{x = (x', x_3) \in \mathbb{R}^3 : x' \in \mathbb{R}^2,\ b(x') < x_3 < \eta_\varepsilon(x')\right\}$.
Let $(u_\varepsilon, p_\varepsilon)$ denotes the unique pair in $(H^1_{\mathrm{per}}(\Omega_\varepsilon))^3 \times L^2(\Omega_\varepsilon)$ satisfying

$$\begin{cases} -\nu \Delta u_\varepsilon + \nabla p_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } P \cup R_\varepsilon, \\ \displaystyle\int_{\Omega^-} p_\varepsilon \, dx = 0, \end{cases}$$

where $f \in (L^2(\Omega))^3$.

Let $(u^-, p^-) \in (H^1_{\rm per}(\Omega^-))^3 \times L^2(\Omega^-)$ the unique solution of

$$\begin{cases} -\nu\Delta u^- + \nabla p^- = f^- & \text{in } \Omega^-, \\ \nabla \cdot u^- = 0 & \text{in } \Omega^-, \\ u^- = 0 & \text{on } \Sigma \cup P, \\ \displaystyle\int_{\Omega^-} p^- \, dx = 0, \end{cases}$$

where $f^- = f_{|\Omega^-}$, then set

$$u = \begin{cases} 0 & \text{in } \Omega^+, \\ u^- & \text{in } \Omega^-. \end{cases}$$

Using classical variational techniques and Bogovski's theorem one can show the following convergence result.

Proposition. Let $\widetilde{u_\varepsilon}$ denote the zero-extension to Ω of u_ε. Then, as $\varepsilon \to 0$.

$$\widetilde{u_\varepsilon} \to u \quad \text{strongly in } (H^1(\Omega))^3,$$

$$p_{\varepsilon\,|\Omega^-} \to p^- \quad \text{strongly in } L^2(\Omega^-).$$

Decay estimates

To construct an asymptotic approximation of the solution $(u_\varepsilon, p_\varepsilon)$ we introduce the solution of a Stokes problem in an infinite vertical domain of $\mathbb{R}^3$.
Let $\Lambda^+ = \widetilde{S} \times (0, +\infty)$, $\Lambda^- = S \times (-\infty, 0)$ and $\Gamma = \widetilde{S} \times \{0\}$

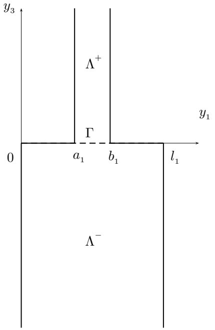

Figure 1: Vertical section of the domain Λ

For $i = 1, 2$, we consider the pairs $(\Psi^{i,+}, \Pi^{i,+})$ and $(\Psi^{i,-}, \Pi^{i,-})$ satisfying

$$\begin{cases} \Psi^{i,+} \in (H^1(\Lambda^+))^3, \ \ \Pi^{i,+} \in L^2_{\text{loc}}(\Lambda^+), \\ \Psi^{i,-} \in (H^1_{\text{loc, per}}(\Lambda^-))^3, \ \ \nabla \Psi^{i,-} \in (L^2(\Lambda^-))^9, \ \ \Pi^{i,-} \in L^2_{\text{loc}}(\Lambda^-), \end{cases}$$

$$\begin{cases} -\nu \Delta \Psi^{i,\pm} + \nabla \Pi^{i,\pm} = 0 \quad \text{in } \Lambda^\pm, \\ \nabla \cdot \Psi^{i,\pm} = 0 \quad \text{in } \Lambda^\pm, \\ \Psi^{i,+} = 0 \quad \text{on } \partial\Lambda^+ \backslash \Gamma, \\ \Psi^{i,-} = 0 \quad \text{on } (S \times \{0\}) \backslash \Gamma, \\ \Psi^{i,+} = \Psi^{i,-} \quad \text{on } \Gamma, \\ \sigma(\Psi^{i,+}, \Pi^{i,+}) n = \sigma(\Psi^{i,-}, \Pi^{i,-}) n + \nu e^i \quad \text{on } \Gamma, \\ \displaystyle\int_{\Lambda^-} \Pi^{i,-} \, dx = 0, \end{cases}$$

where

$$H^1_{\text{loc, per}}(\Lambda^-) = \big\{ v \in H^1(\Lambda') \text{ for any bounded open set } \Lambda' \subset \mathbb{R}^2 \times (-\infty, 0) :$$
$$v(x + (l_1, 0, 0)) = v(x + (0, l_2, 0)) = v(x) \text{ for a.e. } x \in \mathbb{R}^2 \times (-\infty, 0) \big\},$$

$e^1 = (1,0,0)$, $e^2 = (0,1,0)$, $\sigma(\Psi,\Pi) = -\Pi\,\mathrm{I} + 2\nu\, e(\Psi)$, I denoting the 3×3 identity matrix and $e(\Psi) = \frac{1}{2}(\nabla\Psi + (\nabla\Psi)^{\mathsf{T}})$, and n is the unit normal vector on Γ external to Λ_-, i.e. $n = (0,0,1)$.

We denote by β^i the mean of $\Psi^{i,-}$ over a cross section of Λ^- :

$$\beta^i(\delta) = \frac{1}{|S|}\int_S \Psi^{i,-}(y', -\delta)\, dy', \quad \delta \in (0,+\infty),$$

where $y' = (y_1, y_2)$.

Proposition. For each $i = 1,2$, there is a unique solution (Ψ^i, Π^i) of the above Stokes system. Moreover,

(i) the vector β^i is independent of δ, and $\beta^i_3 = 0$;

(ii) for any $\alpha \in \mathbb{N}^3$ and $\delta \in (0,+\infty)$, there exist two positive constants c and $C_{\alpha,\delta}$ such that

$$\left|\partial^\alpha \Psi^{i,+}(y', y_3)\right| + \left|\partial^\alpha \Pi^{i,+}(y', y_3)\right| \le C_{\alpha,\delta}\, e^{-cy_3}, \ \forall (y', y_3) \in \widetilde{S}\times(\delta,+\infty),$$

$$\left|\partial^\alpha (\Psi^{i,-} - \beta^i)(y', y_3)\right| + \left|\partial^\alpha \Pi^{i,-}(y', y_3)\right| \le C_{\alpha,\delta}\, e^{cy_3}, \ \forall (y', y_3) \in S\times(-\infty,-\delta).$$

Asymptotic expansion.

In what follows we assume that

$$b \in H^6_{\mathrm{per}}(S), \quad f_{|\Omega^-} \in (H^4_{\mathrm{per}}(\Omega^-))^3, \quad f_{|\Omega^+} = 0.$$

Let $(w^-, q^-) \in (H^1_{\mathrm{per}}(\Omega^-))^3 \times L^2(\Omega^-)$ be the unique solution of

$$\begin{cases} -\nu \Delta w^- + \nabla q^- = 0 \quad \text{in } \Omega^-, \\ \nabla \cdot w^- = 0 \quad \text{in } \Omega^-, \\ w^- = B \quad \text{on } \Sigma, \\ w^- = 0 \quad \text{on } P, \\ \displaystyle\int_{\Omega^-} q^- \, dx = 0, \end{cases}$$

where

$$B(x') = \sum_{i=1,2} \frac{\partial u_i^-}{\partial x_3}(x', l_3)\, \beta^i, \quad x' \in S,$$

with β^i denotes the mean of $\Psi^{i,-}$ over a cross section of Λ^-.

Let us now set

$$w = \begin{cases} 0 \quad \text{in } \Omega^+, \\ w^- \quad \text{in } \Omega^-, \end{cases} \qquad q = \begin{cases} 0 \quad \text{in } \Omega^+, \\ q^- \quad \text{in } \Omega^-, \end{cases}$$

$$\xi_\varepsilon(x) = \begin{cases} \xi_\varepsilon^+(x) = \sum_{i=1,2} \frac{\partial u_i^-}{\partial x_3}(x', l_3)\, \Psi^{i,+}\left(\frac{x'}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) & \text{in } \Omega_\varepsilon^+, \\ \xi_\varepsilon^-(x) = \sum_{i=1,2} \frac{\partial u_i^-}{\partial x_3}(x', l_3)\, \Psi^{i,-}\left(\frac{x'}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) - B(x') & \text{in } \Omega^-, \end{cases}$$

$$\theta_\varepsilon(x) = \begin{cases} \theta_\varepsilon^+(x) = \sum_{i=1,2} \frac{\partial u_i^-}{\partial x_3}(x', l_3)\, \Pi^{i,+}\left(\frac{x'}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) & \text{in } \Omega_\varepsilon^+, \\ \theta_\varepsilon^-(x) = \sum_{i=1,2} \frac{\partial u_i^-}{\partial x_3}(x', l_3)\, \Pi^{i,-}\left(\frac{x'}{\varepsilon}, \frac{x_3 - l_3}{\varepsilon}\right) & \text{in } \Omega^-, \end{cases}$$

where, for $i = 1, 2$, (Ψ^i, Π^i) is the unique solution of the Stokes system in the domain Λ.

Our first main result is :

Theorem 1. There exists a positive constant C, independent of ε, such that, for any $\gamma > 0$ and ε small enough,

$$\begin{cases} \|u_\varepsilon - u - \varepsilon w - \varepsilon \xi_\varepsilon\|_{(H^1(\Omega_\varepsilon))^3} \le C\varepsilon^{\frac{3}{2}-\gamma}, \\ \left\| p_\varepsilon - p^- - \varepsilon q^- - \left(\theta_\varepsilon^- - \frac{1}{|\Omega^-|}\int_{\Omega^-} \theta_\varepsilon^-\, dx\right)\right\|_{L^2(\Omega^-)} \le C\varepsilon^{\frac{3}{2}-\gamma}. \end{cases}$$

Wall law.

Denote

$$\begin{cases} \mathcal{U}_\varepsilon = u^- + \varepsilon w^- + \varepsilon \xi_\varepsilon^- & \text{in } \Omega^-, \\ \mathcal{P}_\varepsilon = p^- + \varepsilon q^- + \theta_\varepsilon^- & \text{in } \Omega^-. \end{cases}$$

Clearly, $(\mathcal{U}_\varepsilon, \mathcal{P}_\varepsilon) \in (H^1_{\mathrm{per}}(\Omega^-))^3 \times L^2_{\mathrm{per}}(\Omega^-)$ and taking the trace of $\mathcal{U}_\varepsilon$ on $\{x_3 = l_3\}$ we have

$$\mathcal{U}_\varepsilon(x', l_3) = \varepsilon \sum_{i=1,2} \frac{\partial u_i^-}{\partial x_3}(x', l_3) \Psi^{i,-}(y', 0), \quad x' \in S,\ y' = \frac{x'}{\varepsilon}.$$

The fact that $u_3^- = w_3^- = 0$ on $\{x_3 = l_3\}$ provides that

$$\begin{cases} \sigma(u^-, p^-) n\,(x', l_3) = \left(\nu \frac{\partial u_1^-}{\partial x_3}(x', l_3), \nu \frac{\partial u_2^-}{\partial x_3}(x', l_3), -p^-(x', l_3) \right), & x' \in S, \\ \sigma(w^-, q^-) n\,(x', l_3) = \left(\nu \frac{\partial w_1^-}{\partial x_3}(x', l_3), \nu \frac{\partial w_2^-}{\partial x_3}(x', l_3), -q^-(x', l_3) \right), & x' \in S. \end{cases}$$

An easy computation gives

$$\sigma(\varepsilon\xi_\varepsilon^-,\theta_\varepsilon^-)n(x',l_3)=\sum_{i=1,2}\frac{\partial u_i^-}{\partial x_3}(x',l_3)\,\sigma(\Psi^{i,-},\Pi^{i,-})n(y',0)$$

$$+\,\varepsilon\nu\left(\sum_{i=1,2}\frac{\partial}{\partial_{x_1}}\left(\frac{\partial u_i^-}{\partial x_3}(x',l_3)\right)\Psi_3^{i,-}(y',0),\sum_{i=1,2}\frac{\partial}{\partial_{x_2}}\left(\frac{\partial u_i^-}{\partial x_3}(x',l_3)\right)\Psi_3^{i,-}(y',0),0\right)$$

for $x'\in S$, $y'=\frac{x'}{\varepsilon}$.
We now define the mean with respect to $y'\in S$ of a function $\mathcal{U}=\mathcal{U}(x',y')$ by

$$\langle\mathcal{U}\rangle(x')=\frac{1}{|S|}\int_S\mathcal{U}(x',y')\,dy',\ \ x'\in S.$$

Separating the slow and fast variables, taking the mean with respect to $y'\in S$ of $\mathcal{U}_\varepsilon$ and denoting $U_\varepsilon=\langle\mathcal{U}_\varepsilon\rangle$ we obtain

$$U_\varepsilon(x',l_3)=\varepsilon\sum_{i=1,2}\frac{\partial u_i^-}{\partial x_3}(x',l_3)\,\langle\Psi^{i,-}\rangle(0)=\varepsilon B(x'),\ \ x'\in S.$$

Similarly, separating the slow and fast variables, and taking the mean with respect to $y' \in S$, according to the S-periodicity of $\Psi^{i,-}$ and the fact that $\beta_3^i = 0$ we have $\langle \sigma(\Psi^{i,-}, \Pi^{i,-}) n(0) \rangle = \big(0, 0, -\langle \Pi^{i,-} \rangle(0)\big)$ and then

$$\langle \sigma(\varepsilon \xi_\varepsilon^-, \theta_\varepsilon^-) n \rangle (x', l_3) = \big(0, 0, -\langle \theta_\varepsilon^- \rangle (x', l_3)\big), \quad x' \in S.$$

Then, denoting $P_\varepsilon = \langle \mathcal{P}_\varepsilon \rangle$, we deduce that

$$\sigma(U_\varepsilon, P_\varepsilon) n(x', l_3) = \left(\nu \frac{\partial u_1^-}{\partial x_3}(x', l_3), \nu \frac{\partial u_2^-}{\partial x_3}(x', l_3), -p^-(x', l_3) - \langle \theta_\varepsilon^- \rangle (x', l_3) \right)$$
$$+ \varepsilon \left(\nu \frac{\partial w_1^-}{\partial x_3}(x', l_3), \nu \frac{\partial w_2^-}{\partial x_3}(x', l_3), -q^-(x', l_3) \right), \quad x' \in S.$$

Let M denote the 3×3-matrix with column vectors β^1, β^2, 0. Multiplying the previous equality by M yields

$$M \sigma(U_\varepsilon, P_\varepsilon) n(x', l_3) = \nu B(x') + \nu \varepsilon M \frac{\partial w^-}{\partial x_3}(x', l_3), \; x' \in S,$$

then we deduce that

$$\nu U_\varepsilon(x', l_3) - \varepsilon M \sigma(U_\varepsilon, P_\varepsilon) n(x', l_3) = \nu \varepsilon^2 M \frac{\partial w^-}{\partial x_3}(x', l_3), \; x' \in S.$$

Let $\widetilde{M}$ denote the 2×2-matrix with entries $m_{ij} = \beta^i_j$, $1 \le i, j \le 2$ and β^i given by (12). Clearly, for any $v = (\widetilde{v}, v_3) \in \mathbb{R}^3$, with $\widetilde{v} = (v_1, v_2)$, we have $Mv = \left(\widetilde{M}\widetilde{v}, 0\right)$, then one can rewrite the condition on Σ $(\{x_3 = l_3\})$ in the form

$$\nu \widetilde{U}_\varepsilon - \varepsilon \widetilde{M} \frac{\partial \widetilde{U}_\varepsilon}{\partial x_3} = \nu \varepsilon^2 \widetilde{M} \frac{\partial \widetilde{w}^-}{\partial x_3} \quad \text{on } \Sigma, \qquad U_{\varepsilon 3} = 0 \quad \text{on } \Sigma.$$

Neglecting the ε^2-term in the previous relation we derive the wall law

$$\nu \widetilde{U}_\varepsilon - \varepsilon \widetilde{M} \frac{\partial \widetilde{U}_\varepsilon}{\partial x_3} = 0 \quad \text{on } \Sigma, \qquad U_{\varepsilon 3} = 0 \quad \text{on } \Sigma.$$

Note also that the previous boundary condition is equivalent to the following one

$$\nu U_\varepsilon - \varepsilon M \frac{\partial U_\varepsilon}{\partial x_3} = 0 \quad \text{on } \Sigma.$$

Lemma. The matrix $\widetilde{M}$ is symmetric and negative definite.

Consider the system

$$\begin{cases} -\nu \Delta U_\varepsilon + \nabla P_\varepsilon = f \quad \text{in } \Omega^-, \\ \nabla \cdot U_\varepsilon = 0 \quad \text{in } \Omega^-, \\ U_\varepsilon - \varepsilon M \frac{\partial U_\varepsilon}{\partial x_3} = 0 \quad \text{on } \Sigma, \\ U_\varepsilon = 0 \quad \text{on } P, \\ \displaystyle\int_{\Omega^-} P_\varepsilon^- \, dx = 0. \end{cases} \tag{2}$$

According to the property of the matrix $\widetilde{M}$ one can show the following result.

Lemma. Problem (2) has a unique solution $(U_\varepsilon, P_\varepsilon) \in H^1_{\mathrm{per}}(\Omega^-))^3 \times L^2(\Omega^-)$.

Our second main result is :

Theorem 2. Let $(u_\varepsilon, p_\varepsilon)$ be the solution of the original Stokes system and let $(U_\varepsilon, P_\varepsilon)$ be the solution of (2). Then, there exists a positive constant C, independent of ε, such that, for any $\gamma > 0$ and ε small enough,

$$\begin{cases} \|u_\varepsilon - U_\varepsilon - \varepsilon \xi_\varepsilon\|_{(H^1(\Omega^-))^3} \leq C\varepsilon^{\frac{3}{2}-\gamma}, \\ \left\| p_\varepsilon - P_\varepsilon - \left(\theta_\varepsilon^- - \frac{1}{|\Omega^-|}\int_{\Omega^-} \theta_\varepsilon^- \, dx\right)\right\|_{L^2(\Omega^-)} \leq C\varepsilon^{\frac{3}{2}-\gamma}. \end{cases}$$

Sketch of the proof of Theorem 2.

Let $(\varphi_\varepsilon, \mu_\varepsilon)$ be defined by

$$\begin{cases} \varphi_\varepsilon = u^- + \varepsilon w^- - U_\varepsilon & \text{in } \Omega^-, \\ \pi_\varepsilon = p^- + \varepsilon q^- - P_\varepsilon & \text{in } \Omega^-. \end{cases}$$

We easily verify that $(\varphi_\varepsilon, \pi_\varepsilon) \in (H^1_{\rm per}(\Omega^-))^3 \times L^2(\Omega^-)$ and satisfies

$$\begin{cases} -\nu\Delta\varphi_\varepsilon + \nabla\pi_\varepsilon = 0 \quad \text{in } \Omega^-, \\ \nabla\cdot\varphi_\varepsilon = 0 \quad \text{in } \Omega^-, \\ \widetilde{\varphi}_\varepsilon - \varepsilon\widetilde{M}\frac{\partial\widetilde{\varphi_\varepsilon}}{\partial x_3} = -\varepsilon^2\widetilde{M}\frac{\partial\widetilde{w^-}}{\partial x_3} \quad \text{on } \Sigma, \\ \varphi_{\varepsilon 3} = 0 \quad \text{on } \Sigma \\ \varphi_\varepsilon = 0 \quad \text{on } P, \\ \displaystyle\int_{\Omega^-} \pi_\varepsilon \, dx = 0. \end{cases}$$

The variational formulation of this problem reads : Find $\varphi_\varepsilon \in W(\Omega^-)$ such that

$$2\nu\int_{\Omega^-} e(\varphi_\varepsilon) : e(\varphi)\, dx + \frac{\nu}{\varepsilon}\int_\Sigma \left(N\widetilde{\varphi_\varepsilon}\right)\cdot\widetilde{\varphi}\, ds = \nu\varepsilon\int_\Sigma \frac{\partial\widetilde{w^-}}{\partial x_3}\cdot\widetilde{\varphi}\, ds, \quad \forall\varphi\in W(\Omega^-).$$

Here

$$W(\Omega^-) = \left\{\varphi\in(H^1_{\rm per}(\Omega^-))^3 : \nabla\cdot\varphi = 0 \ \text{ in } \Omega^-,\ \varphi = 0 \ \text{ on } P,\ \varphi_3 = 0 \ \text{ on } \Sigma\right\}.$$

Taking $\varphi = \varphi_\varepsilon$ in the previous variational formulation, using the fact that the matrix N is positive definite, we show that there exists a positive constant C such that

$$\frac{C}{\varepsilon}\int_\Sigma |\widetilde{\varphi_\varepsilon}|^2\, ds \leq \frac{\nu}{\varepsilon}\int_\Sigma \left(N\widetilde{\varphi_\varepsilon}\right)\cdot \widetilde{\varphi}_\varepsilon\, ds \leq C\varepsilon \|\widetilde{\varphi_\varepsilon}\|_{(L^2(\Sigma))^3}$$

therefore

$$\|\widetilde{\varphi}_\varepsilon\|_{(L^2(\Sigma))^3} \leq C\varepsilon^2.$$

Then, using the Korn, we deduce that

$$\|\varphi_\varepsilon\|_{(H^1(\Omega-))^3} \leq C\varepsilon^{3/2}.$$

Since $\int_{\Omega^-} \pi_\varepsilon\, dx = 0$, there exists $\rho_\varepsilon \in (H_0^1(\Omega^-))^3$ such that $\nabla\cdot\rho_\varepsilon = \pi_\varepsilon$ in Ω^- and $\|\rho_\varepsilon\|_{(H_0^1(\Omega-))^3} \leq C\|\pi_\varepsilon\|_{L^2(\Omega-)}$. Then, considering the equation satisfied by $(\varphi_\varepsilon, \pi_\varepsilon)$ we deduce that

$$\|\pi_\varepsilon\|_{L^2(\Omega-)} \leq C\varepsilon^{3/2}.$$

Now, writing $\tau_\varepsilon^0 = u_\varepsilon - u^- - \varepsilon w^- - \varepsilon \xi_\varepsilon^-$ and $\mu_\varepsilon^0 = p_\varepsilon - p^- - \varepsilon q^- - (\theta_\varepsilon^- - d_\varepsilon)$, where $d_\varepsilon = \frac{1}{|\Omega^-|_\varepsilon} \int_{\Omega^-} \theta_\varepsilon^- \, dx$, we have

$$
\begin{aligned}
u_\varepsilon - U_\varepsilon - \varepsilon \xi_\varepsilon^- &= \tau_\varepsilon^0 + \varphi_\varepsilon, \\
p_\varepsilon - P_\varepsilon - (\theta_\varepsilon^- - d_\varepsilon) &= \mu_\varepsilon^0 + \pi_\varepsilon,
\end{aligned}
$$

and estimates in Theorem 2 follow from that in Theorem 1. □

Sketch of the proof of Theorem 1.

We introduce the system

$$
\begin{cases}
-\nu \Delta w_\varepsilon^+ + \nabla q_\varepsilon^+ = 0 \quad \text{in } \Omega_\varepsilon^+, \\
-\nu \Delta w_\varepsilon^- + \nabla q_\varepsilon^- = 0 \quad \text{in } \Omega^-, \\
\nabla \cdot w_\varepsilon^+ = -\nabla \cdot \xi_\varepsilon^+ \quad \text{in } \Omega_\varepsilon^+, \\
\nabla \cdot w_\varepsilon^- = -\nabla \cdot \xi_\varepsilon^- \quad \text{in } \Omega^-, \\
w_\varepsilon^+ = -\xi_\varepsilon^+ \quad \text{on } R_\varepsilon \backslash \Sigma, \\
w_\varepsilon^- = B \quad \text{on } R_\varepsilon \cap \Sigma, \\
w_\varepsilon^- = -\xi_\varepsilon^- \quad \text{on } P, \\
w_\varepsilon^+ = w_\varepsilon^- - B \quad \text{on } \Sigma \backslash R_\varepsilon, \\
\sigma(w_\varepsilon^+, q_\varepsilon^+) n = \sigma(w_\varepsilon^-, q_\varepsilon^-) n - \frac{1}{\varepsilon} \sigma(0, p^-) n \quad \text{on } \Sigma \backslash R_\varepsilon,
\end{cases}
$$

where $B(x') = \sum_{i=1,2} \frac{\partial u_i^-}{\partial x_3}(x', l_3)\, \beta^i, \;\; x' \in S$ and $n = (0, 0, 1)$.

Let τ_ε and μ_ε be defined by

$$\tau_\varepsilon = \begin{cases} \tau_\varepsilon^+ = u_\varepsilon - \varepsilon w_\varepsilon^+ - \varepsilon \xi_\varepsilon^+ & \text{in } \Omega_\varepsilon^+, \\ \tau_\varepsilon^- = u_\varepsilon - u^- - \varepsilon w_\varepsilon^- - \varepsilon \xi_\varepsilon^- & \text{in } \Omega^-, \end{cases}$$

$$\mu_\varepsilon = \begin{cases} \mu_\varepsilon^+ = p_\varepsilon - \varepsilon q_\varepsilon^+ - \theta_\varepsilon^+ & \text{in } \Omega_\varepsilon^+, \\ \mu_\varepsilon^- = p_\varepsilon - p^- - \varepsilon q_\varepsilon^- - \theta_\varepsilon^- & \text{in } \Omega^-. \end{cases}$$

We impose

$$\int_{\Omega^-} q_\varepsilon^-(x)\, dx = -\frac{1}{\varepsilon} \int_{\Omega^-} \theta_\varepsilon^-(x)\, dx$$

so that $\int_{\Omega^-} \mu_\varepsilon^-(x)\, dx = 0$.

The proof of Theorem 1 consists in three steps

Step 1 : : Estimate of τ_ε and μ_ε

Proposition. There exists a positive constant C, independent of ε, such that, for any $\gamma > 0$ and ε small enough,

$$\|\tau_\varepsilon\|_{(H^1(\Omega_\varepsilon))^3} + \|\mu_\varepsilon\|_{L^2(\Omega^-)} \leq C\varepsilon^{\frac{3}{2}-\gamma}.$$

We prove this result by writing the Stokes system verified by $(\tau_\varepsilon, \mu_\varepsilon)$ in the domain Ω_ε (without interface conditions).

Then we note that

$$u_\varepsilon - u - \varepsilon w - \varepsilon \xi_\varepsilon = \tau_\varepsilon - \varepsilon(w_\varepsilon - w)$$

$$p_\varepsilon - p^- - \varepsilon q^- - (\theta_\varepsilon^- - d_\varepsilon) = \mu_\varepsilon - \varepsilon(q_\varepsilon - q + \frac{d_\varepsilon}{\varepsilon}),$$

where $d_\varepsilon = \frac{1}{|\Omega^-|}\int_{\Omega^-} \theta_\varepsilon^- \, dx$. Therefore to prove Theorem 1 we have to estimate $\|w_\varepsilon - w\|_{(H^1(\Omega_\varepsilon))^3}$ and $q_\varepsilon - q + \frac{d_\varepsilon}{\varepsilon}$.

Step 2 : Estimate of $\mathbf{w}_\varepsilon - \mathbf{w}$

Proposition. There is a positive constant C, independent of ε, such that, for any $\gamma > 0$ and ε small enough,

$$\|w_\varepsilon - w\|_{(H^1(\Omega_\varepsilon))^3} \leq C\varepsilon^{\frac{1}{2}-\gamma}.$$

Step 3 : Estimate of $\mathbf{q}_\varepsilon - \mathbf{q}$

Proposition. There is a positive constant C, independent of ε, such that, for any $\gamma > 0$ and ε small enough,

$$\left\| q_\varepsilon - q + \frac{d_\varepsilon}{\varepsilon} \right\|_{L^2(\Omega^-)} \leq C\varepsilon^{\frac{1}{2}-\gamma}.$$

To prove this proposition we consider the decomposition

$$w_\varepsilon - w = V_\varepsilon + V_\varepsilon^0 + W_\varepsilon, \quad q_\varepsilon - q + \frac{d_\varepsilon}{\varepsilon} = r_\varepsilon + r_\varepsilon^0 + Q_\varepsilon.$$

where :

– the pair $(V_\varepsilon, r_\varepsilon) \in (H^1_{\mathrm{per}}(\Omega_\varepsilon))^3 \times L^2(\Omega_\varepsilon)$ is the solution of the system

$$\begin{cases} -\nu \Delta V_\varepsilon^+ + \nabla r_\varepsilon^+ = 0 \quad \text{in } \Omega_\varepsilon^+, \\ -\nu \Delta V_\varepsilon^- + \nabla r_\varepsilon^- = 0 \quad \text{in } \Omega^-, \\ \nabla \cdot V_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ V_\varepsilon = 0 \quad \text{on } P \cup R_\varepsilon, \\ \sigma(V_\varepsilon^+, r_\varepsilon^+)n = \sigma(V_\varepsilon^-, r_\varepsilon^-)n + \sigma(w^-, q^-)n \quad \text{on } \Sigma \backslash R_\varepsilon, \\ \int_{\Omega^-} r_\varepsilon^-(x)\, dx = 0; \end{cases}$$

– the pair $(V_\varepsilon^0, r_\varepsilon^0) \in (H^1_{\mathrm{per}}(\Omega_\varepsilon))^3 \times L^2(\Omega_\varepsilon)$ is the solution of the system

$$\begin{cases} -\nu \Delta V_\varepsilon^{0,+} + \nabla r_\varepsilon^{0,+} = 0 \quad \text{in } \Omega_\varepsilon^+, \\ -\nu \Delta V_\varepsilon^{0,-} + \nabla r_\varepsilon^{0,-} = 0 \quad \text{in } \Omega^-, \\ \nabla \cdot V_\varepsilon^0 = 0 \quad \text{in } \Omega_\varepsilon, \\ V_\varepsilon^0 = 0 \quad \text{on } P \cup R_\varepsilon, \\ \sigma(V_\varepsilon^{0,+}, r_\varepsilon^{0,+})n = \sigma(V_\varepsilon^{0,-}, r_\varepsilon^{0,-})n - \frac{1}{\varepsilon}\sigma(0, p^-)n \quad \text{on } \Sigma \backslash R_\varepsilon, \\ \int_{\Omega^-} r_\varepsilon^{0,-}(x)\, dx = 0; \end{cases}$$

– the pair $(W_\varepsilon, Q_\varepsilon) \in (H^1_{\mathrm{per}}(\Omega_\varepsilon))^3 \times L^2(\Omega_\varepsilon)$ is the solution of the system

$$\begin{cases} -\nu \Delta W_\varepsilon + \nabla Q_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ \nabla \cdot W_\varepsilon = -\nabla \cdot \xi_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ W_\varepsilon^+ = -\xi_\varepsilon^+ \quad \text{on } R_\varepsilon \backslash \Sigma, \\ W_\varepsilon^- = -\xi_\varepsilon^- \quad \text{on } P, \\ W_\varepsilon^- = 0 \quad \text{on } R_\varepsilon \cap \Sigma, \\ \int_{\Omega^-} Q_\varepsilon^-(x)\, dx = 0. \end{cases}$$

We show that

$$\|V_\varepsilon\|_{(H^1(\Omega_\varepsilon))^3} + \|V_\varepsilon^0\|_{(H^1(\Omega_\varepsilon))^3} \leq C\sqrt{\varepsilon},$$

$$\|W_\varepsilon\|_{(H^1(\Omega_\varepsilon))^3} \leq C\varepsilon^{\frac{1}{2}-\gamma},$$

then, applying Bogovski's theorem and using the previous inequalities we prove that

$$\left\| q_\varepsilon - q^- + \frac{d_\varepsilon}{\varepsilon} \right\|_{L^2(\Omega^-)} \leq C\varepsilon^{\frac{1}{2}-\gamma}.$$

This completes the proof of Theorem 1. □